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Exact Results for Correlations in a Two-Component Log-Gas

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A study of the one-dimensional lattice gas of positive and negative charges interacting via the logarithmic potential is continued. The two-particle distribution functions are evaluated exactly at the couplings $\Gamma = 2$ and 4. It is proved that the $\Gamma = 4$ isotherm exhibits an insulator–conductor phase transition at the reduced density 1/2, and the scaling behavior of the correlations near this critical point is given. Similarities of the conjectured phase diagram with that of a one-dimensional one-component log-gas in a periodic potential are noted.

KEY WORDS: Insulator–conductor phase transition; two-component plasma; exact solution.

1. INTRODUCTION

In a previous paper,$^{(1)}$ to be referred to as I, exact calculations of the equilibrium statistical mechanics of a one-dimensional lattice gas of opposite charges interacting via a logarithmic potential were performed. For the isotherms $\Gamma = 2$ and 4 [see (1.3) below for the definition of Γ], the polynomial defining the grand partition function Ξ was factorized for each value of M (the total number of lattice sites available to each charge species). The pressure and density were then obtained as functions of the fugacity in the thermodynamic limit.

This particular log-gas system is of immediate physical interest for its relationship to the quantum Brownian motion problem in a periodic potential.$^{(2)}$ Explicitly, the grand partition function of the log-gas, in the low-density limit at least, is identical to the generating functional of the quantum problem. Furthermore, the mobility $\mu(\omega)$ in the quantum problem is (up to a simple factor) equal to the dielectric constant $\varepsilon(k)$ of

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the log-gas (k is to be identified with ω). The dielectric constant can be defined by

$$1/\varepsilon(k) = 1 - [\hat{C}^T(k)/q^2][\beta\hat{\phi}(k)] \tag{1.1}$$

where $\hat{C}^T(k)$ denotes the Fourier transform of the truncated charge–charge distribution $C(y)$, q denotes the unit of charge, and $\beta\hat{\phi}(k)$ is the Fourier transform of the dimensionless potential. Explicitly,

$$C(y) = 2q^2[\rho^T_{+,+}(y) - \rho^T_{+,-}(y)] \tag{1.2}$$

where $\rho^T_{+,+}(y)$ denotes the truncated distribution function for like charges and $\rho^T_{+,-}(y)$ the same for unlike charges. Also,

$$\beta\phi(y) = -\Gamma \log|y|, \qquad \Gamma = q^2/k_B T \tag{1.3}$$

so that

$$\beta\hat{\phi}(k) = \Gamma\pi/|k| \tag{1.4}$$

In the low-density regime, the log-gas consists of two phases$^{(3,4)}$: a high-temperature conducting phase in which $\varepsilon(k) \to \infty$ as $k \to 0$, and a low-temperature dielectric phase in which $\varepsilon(k) \to 1$ as $k \to 0$ (it is possible to argue from electrostatics$^{(5)}$ that a one-dimensional region of two-dimensional dielectric material behaves as a vacuum). From (1.1) and (1.4) the conducting phase is thus characterized by the behavior

$$\hat{C}^T(k)/q^2 \sim |k|/\pi\Gamma \qquad \text{as} \quad k \to 0 \tag{1.5}$$

while for the dielectric phase

$$\hat{C}^T(k) \sim o(|k|) \tag{1.6}$$

The exact calculations performed in I revealed new features of the system worthy of study. First, I was led to conjecture a remarkable mathematical mechanism underlying the conducting–insulating phase transition: all the zeros of the grand partition function collapse on the point $\xi = -1$ in the scaled complex fugacity plane ξ. More explicitly, I conjecture that all the zeros of Ξ (for any fixed number M of lattice sites) lie on the negative real axis for $\Gamma < 2$, on the point $\xi = -1$ for $\Gamma = 2$, and on the unit circle in the complex ξ plane for $\Gamma > 2$. These claims were proved for all M at $\Gamma = 2$ and 4 and for all Γ with $M = 2$, 3, and 4.

Furthermore, the exact calculations revealed a phase transition along the $\Gamma = 4$ isotherm at the reduced density $\tau\rho_+ = \tau\rho_- = 1/2$ (every second lattice site is occupied), where τ denotes the lattice spacing while ρ_+ and

ρ_- denote the particle density of positive and negative charges, respectively. Mathematically, in accordance with the Yang–Lee(6) theory, this is a consequence of the zeros of the grand partition function pinching the real axis in the thermodynamic limit. I conjectured that this phase transition persists for all $\Gamma > 2$.

In this paper I pursue exact calculations by evaluating the two-particle distribution functions at $\Gamma = 2$ and 4. In the latter case, this makes it possible to categorize the phase at $\Gamma = 4$ and $\tau\rho_+ = 1/2$ as conducting and explicitly calculate the behavior of the correlation length near the critical point. At $\Gamma = 2$ one again encounters the remarkable "conformal invariance" property of the truncated n-particle distribution function for like particles at nonzero separation from each other: they vanish identically.(7)

2. DISTRIBUTION FUNCTIONS AT Γ = 2

Let us first recall the precise definition of the system. Divide a line of length L into M intervals so that there are sites at the points nL/N, $n = 1, 2,..., M$. Introduce an interlacing lattice at the points $(n - 1/2)\, L/M$, $n = 1, 2,..., m$. Denote these lattices $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively. Let the positive charges occupy $\mathscr{L}_1$ and denote the coordinates by $m_k L/M$, and let the negative charges occupy $\mathscr{L}_2$ with the coordinates given by $(l_k - 1/2)\, L/M$. Further impose periodic boundary conditions so that the pair potential is

$$V(\theta_1, \theta_2) = -q_1 q_2 \log\{|e^{2\pi i\theta_1/L} - e^{2\pi i\theta_2/L}|\, (L/2\pi)\} \tag{2.1}$$

2.1. *n*-Particle Distribution Function for Like Charges

In the grand canonical ensemble, the dimensionless n-particle distribution for n charges of like sign (negative, say) at $l'_1 - 1/2$, $l'_2 - 1/2$,..., $l'_n - 1/2$ is given by

$$\rho_{n(-)}\left(l'_1 - \frac{1}{2},..., l'_n - \frac{1}{2}\right) = \frac{1}{\Xi_\Gamma} \sum_{N=0}^{M} \zeta^{2N} Z_N(l'_1, l'_2,..., l'_n) \tag{2.2}$$

where

$$Z_N(l'_1,..., l'_n) = \frac{\delta^n}{\delta a(l'_1) \cdots \delta a(l'_n)} \times \sum_{l_1,...,l_N = 1}^{M} \sum_{m_1,...,m_N = 1}^{M} \prod_{j=1}^{N} [1 + a(l_j)]\, W_{N\Gamma}/(N!)^2 \Big|_{a=0} \tag{2.3}$$

Here Ξ_Γ denotes the grand partition function, ζ denotes the fugacity, $W_{N\Gamma}$ denotes the Boltzmann factor, and $\delta/\delta a(k)$ denotes functional differentiation.

In ref. 1, I derived the result

$$\Xi_2 = (1+\xi)^M \tag{2.4}$$

where ξ, the scaled fugacity, is defined for general Γ by

$$\xi = (\pi M/L)^\Gamma \zeta^2 \tag{2.5}$$

Furthermore, previous work$^{(1)}$ gives the formula

$$\begin{aligned} Z_N(l'_1,\ldots,l'_n) &= (N-n)^{-1}\,(\pi/L)^{2N}\,M^N \\ &\quad\times \sum_{l_{n+1},\ldots,l_N=1}^{M}\ \sum_{0\leqslant\gamma_1,\ldots,\gamma_N}^{M-1} \left(\prod_{k=1}^{N}\delta_{\gamma_k,\nu_k}\right)\left(\prod_{k=1}^{n}\delta_{l_k,l'_k}\right) \\ &\quad\times \operatorname{Det}\left[e^{-2\pi i(l_k-1/2)(\gamma_j-\nu_k)/M}\right]_{j,k=1,\ldots,N} \end{aligned} \tag{2.6}$$

The determinant in (2.6) vanishes if $\gamma_k=\gamma_{k'}$ for $k\neq k'$, so one can make the restriction $\gamma_k\neq\gamma_{k'}$. Also, the summand in (2.6) is symmetrical in $\gamma_{n+1},\ldots,\gamma_N$. Thus, one can write

$$\sum_{0\leqslant\gamma_1,\ldots,\gamma_N}^{M-1} = (N-n)!\sum_{0\leqslant\gamma_1,\ldots,\gamma_n}^{M-1}\ \sum_{\substack{0\leqslant\gamma_{n+1}<\gamma_{n+2}<\cdots<\gamma_N\leqslant M-1\\ \neq\gamma_1,\ldots,\gamma_n}} \tag{2.7}$$

The sum over the l's in (2.6) can be performed row by row in the determinant [beginning with the $(n+1)$th row]. Due to (2.7), the only nonzero contribution is from the diagonal terms. After some straightforward manipulation of the resulting elements of the determinant we obtain

$$\begin{aligned} Z_N(l'_1,\ldots,l'_n) &= \left(\frac{\pi}{L}\right)^{2N} M^{2N-n}\sum_{0\leqslant\gamma_1,\ldots,\gamma_n}^{M-1}\operatorname{Det}\left[\exp\frac{-2\pi i\gamma_k(l'_k-l'_j)}{M}\right]_{j,k=1,\ldots,n} \\ &\quad\times\sum_{\substack{0\leqslant\gamma_{n+1}<\cdots<\gamma_N\leqslant M-1\\ \neq\gamma_1,\ldots,\gamma_n}} 1 \end{aligned} \tag{2.8}$$

The second sum in (2.8) is independent of the particular choice of $\gamma_1,\ldots,\gamma_n$, so one can choose $\gamma_j=j-1$, $j=1,\ldots,n$. Now the sum over $\gamma_1,\ldots,\gamma_n$ can be performed column by column. Unless $l'_k=l'_{k'}$ for $k=k'$ [in which case (2.8) vanishes], the only nonzero terms are the diagonals (each giving M), so one has the remarkable result that the function (2.3) is given by

$$\zeta^N Z_N(l'_1,\ldots,l'_n) = \xi^N\binom{M-n}{N-n}\prod_{1\leqslant j<k\leqslant n}(1-\delta_{l'_j,l'_k}) \tag{2.9}$$

Substituting (2.9) in (2.2), one obtains

$$\begin{aligned}\rho_n(l'_1,\dots,l'_n) &= \left(\frac{\xi}{1+\xi}\right)^n \prod_{1\leqslant j<k\leqslant n}(1-\delta_{l'_j,l'_k}) \\ &= (\tau\rho_-)^n \prod_{1\leqslant j<k\leqslant n}(1-\delta_{l'_j,l'_k}) \qquad (2.10)\end{aligned}$$

where τ and ρ_- have been defined in the Introduction. In the thermodynamic limit, the result (2.10) has been obtained previously.[7]

2.2. Distribution Function for Opposite Charges

In the grand canonical ensemble, the dimensionless distribution function for a positive charge at m_a and a negative charge at $l_a - 1/2$ is given by

$$\rho_{+,-}\left(m_a, l_a - \frac{1}{2}\right) = \frac{1}{\Xi_\Gamma}\sum_{N=0}^{M}\zeta^{2N}Z_N\left(m_a, l_a-\frac{1}{2}\right) \qquad (2.11)$$

where

$$Z_N\left(m_a, l_a-\frac{1}{2}\right) = \frac{\delta^2}{\delta a(m_a)\,\delta b(l_a)} \sum_{l_1,\dots,l_N=1}^{M}\;\sum_{m_1,\dots,m_N=1}^{M}\;\prod_{k=1}^{N}[1+a(m_k)][1+b(l_k)]\,\frac{W_{N\Gamma}}{(N!)^2}\bigg|_{a=b=0} \qquad (2.12)$$

From the workings of I, one can readily show

$$\begin{aligned}\zeta^{2N}Z_N(m_a, l_a-1/2) &= \xi^N[(N-1)!]^{-1}M^{-2} \\ &\times \sum_{\gamma_1,\dots,\gamma_N=0}^{M-1}\;\sum_{\nu_p=0}^{M-1}\;\sum_{p=1}^{N}\left(\prod_{\substack{k=1\\k\neq p}}^{N}\delta_{\gamma_k,\nu_k}\right)e^{2\pi i m_a(\gamma_p-\nu_p)/M} \\ &\times \mathrm{Det}[d_{jk}]_{j,k=1,\dots,N} \qquad (2.13)\end{aligned}$$

The elements of the determinant in (2.13) are

$$d_{jk} = \begin{cases} e^{-2\pi i(l_a-1/2)(\gamma_1-\nu_k)/M}, & j=1, \quad k=1,2,\dots,N \\ \delta_{\gamma_j,\nu_k}, & j\geqslant 2, \quad k=1,2,\dots,N \end{cases} \qquad (2.14)$$

Taking into account the Kronecker deltas in (2.13), the determinant vanishes if $\gamma_k=\gamma_{k'}$ for $k\neq k'$ $(k, k'\geqslant 1)$ or $\nu_1=\gamma_k$ $(k\geqslant 2)$, so one can make the restriction $\gamma_k\neq\gamma_{k'}$ and $\nu_1\neq\gamma_k$.

By considering each term in the sum over p in (2.13) individually, we see that the $p=1$ term gives a different contribution to the $p \geqslant 2$ cases. Separating the two types of terms and ordering the γ's gives

$$\begin{aligned}
&\zeta^{2N} Z_N(m_a, l_a - 1/2) \\
&\quad = \xi^N M^{-2} \Bigg\{ \sum_{\gamma_1=0}^{M-1} \sum_{\gamma_2=0}^{M-1} \sum_{\substack{0 \leqslant \gamma_3 < \cdots < \gamma_N \leqslant M-1 \\ \neq \gamma_1, \gamma_2}} 1 \\
&\qquad + \sum_{\gamma_1=0}^{M-1} \sum_{\nu_1=0}^{M-1} e^{2\pi i (m_a - l_a + 1/2)(\gamma_1 - \nu_1)/M} \sum_{\substack{0 \leqslant \gamma_2 < \cdots < \gamma_N \leqslant M-1 \\ \neq \gamma_1, \nu_1}} 1 \Bigg\} \qquad (2.15)
\end{aligned}$$

Using the same argument which led to (2.9), this reduces to

$$\begin{aligned}
&\zeta^{2N} Z_N\left(m_a, l_a - \frac{1}{2}\right) \\
&\quad = \xi^N \left\{ \binom{M-2}{N-2} + \frac{1}{M^2 \sin^2 \pi (m_a - l_a + 1/2)/M} \binom{M-2}{N-1} \right\} \qquad (2.16)
\end{aligned}$$

Substituting (2.16) and (2.4) in (2.11) then gives

$$\rho_{+,-}^T \left(m_a, l_a - \frac{1}{2}\right) = \frac{\tau \rho (1 - \tau \rho)}{M^2 \sin^2 \pi (m_a - l_a + 1/2)/M} \qquad (2.17)$$

where $\rho = \rho_+ = \rho_-$ is the particle density of either species. In particular, one has in the thermodynamic limit

$$\rho_{+,-}^T \left(m_a, l_a - \frac{1}{2}\right) = \frac{\tau \rho (1 - \tau \rho)}{\pi^2 (m_a - l_a + 1/2)^2} \qquad (2.18)$$

which agrees with a result of Gaudin.(7)

3. TWO-PARTICLE DISTRIBUTION FUNCTIONS AT Γ=4

In ref. 1, I showed that the Boltzmann factor for N positive and N negative charges at $\Gamma = 4$ can be written

$$\begin{aligned}
W_{N4} = {} & 2^{-2N} (2\pi/L)^{4N} \sum_{\gamma} \prod_{k=1}^{N} z_k^{\gamma_{2k} + \gamma_{2k-1} + 1} \\
& \times \sum_{X} \varepsilon(P) \prod_{k=1}^{N} w_k^{-\gamma_{P(2k-1)} - \gamma_{P(2k)} - 1} f_{P(2k-1), P(2k)} \qquad (3.1)
\end{aligned}$$

where

$$\begin{aligned} f_{P(2k-1),P(2k)} &= a_{\chi(P(2k))}(\gamma_{P(2k)})[a_1(\gamma_{P(2k-1)})-1]^{\chi(P(2k-1))-1} \\ &\quad - a_{\chi(P(2k-1))}(\gamma_{P(2k-1)})[a_1(\gamma_{P(2k)})-1]^{\chi(P(2k))-1} \\ w_k &= e^{2\pi i m_k/M}, \qquad z_k = e^{2\pi i(l_k-1/2)/M} \\ a_1(\gamma) &= \gamma+1-M/2, \qquad a_2(\gamma)=\gamma^2+(1-M)\gamma-M/2 \\ \chi(k) &= \begin{cases} 1, & k \text{ odd} \\ 2, & k \text{ even} \end{cases} \end{aligned} \tag{3.2}$$

γ denotes the summation range

$$0 \leqslant \gamma_1, \gamma_2, ..., \gamma_{2N} \leqslant M-1, \qquad \gamma_{2j} \neq \gamma_{2j'}, \qquad \gamma_{2j-1} \neq \gamma_{2j'-1}$$

and

$$X = \{P\colon P(2l) > P(2l-1) \text{ each } l=1,2,..., N\}$$

This expression was used to evaluate the grand canonical partition function Ξ_4 as

$$\begin{aligned} \Xi_4 &= \prod_{l=0}^{[M/2]-1} \left\{1-\xi\frac{4}{M^2}\left[4l^2-4l(M-1)+\frac{M^2}{2}-2M+1\right]+\xi^2\right\} \\ &\quad \times \begin{cases} 1, & M \text{ even} \\ 1+\xi, & M \text{ odd} \end{cases} \end{aligned} \tag{3.3}$$

where ξ is given by (2.5) with $\Gamma=4$.

3.1. Like Charges

It is most convenient in this case to calculate the two-particle distribution function for two positive charges, which are supposed fixed at m_1 and m_2. Use the definition [equivalent to (2.2)]

$$\rho_{+,+}(m_1, m_2) = \frac{1}{\Xi_4} \sum_{N=2}^{M} \zeta^{2N} Z_N(m_1, m_2) \tag{3.4}$$

where

$$Z_N(m_1, m_2) = \sum_{m_3,...,m_N=1}^{M} \sum_{l_1,...,l_N=1}^{M} W_{N4}/[N!\,(N-2)!] \tag{3.5}$$

Substituting (3.1) in (3.5) allows the sum over $l_1,\dots, l_N$ and $m_3,\dots, m_N$ to be performed. This gives

$$(-1)^N M^{2N-2} \prod_{k=1}^{N} \delta_{\gamma_{2k}+\gamma_{2k-1}+1,M} \prod_{j=3}^{N} \delta_{P(2j)+P(2j-1)+1,M} \tag{3.6}$$

To proceed further, it is necessary to list permissible choices [i.e., those consistent with the constraints given in (3.2)] of $P(1),\dots, P(4)$. The permissible choices can be divided into different classes. These classes can be categorized according to the leading-order ξ behavior in a low-density expansion, which is determined by the number of permutations directly affected by the particular choice of $P(1),\dots, P(4)$. There are three possibilities, ξ^2, ξ^3, or ξ^4, which correspond to directly affecting no other permutations; $P(5)$, $P(6)$; and $P(5),\dots, P(8)$, respectively. To illustrate the reasoning necessary to evaluate the contribution to (3.5) for each class, I outline the working for the ξ^2 class.

The ξ^3 class specifies $P(1),\dots, P(6)$, which I list in Table I, together with the corresponding value of $\varepsilon(P)$ and the multiplicity.

Here $s_1 < s_2 < s_3$, and also to be included in each contribution are the interchanges $(P(1), P(2)) \leftrightarrow (P(3), P(4))$. The column headed $\varepsilon(P)$ gives the value of the permutation assuming $P(2l-1) = 2r_l - 1$ and $P(2l) = 2r_l$ for all $l \geqslant 4$. The multiplicity column gives the number of possible permutations of s_1, s_2, s_3 which give the same contribution to (3.5).

First, note that we could have specified any of $(P(2l-1), P(2l))$, $l \geqslant 3$, instead of $(P(5), P(6))$. But all such choices give the same contribution to (3.5), so we have a factor of

$$N-2 \tag{3.7}$$

Further, we can choose $s_j = j$, $j = 1, 2, 3$, provided we multiply by

$$\sum_{1 \leqslant s_1 < s_2 < s_3 \leqslant N} 1 = \binom{N}{3} \tag{3.8}$$

Table I. The ξ^3 Class of Permutations for $\rho_{+,+}$

	$P(1)$	$P(2)$	$P(3)$	$P(4)$	$P(5)$	$P(6)$	$\varepsilon(P)$	Multiplicity
(i)	$2s_1-1$	$2s_1$	$2s_2$	$2s_3$	$2s_2-1$	$2s_3-1$	-1	3
(ii)	$2s_1-1$	$2s_1$	$2s_2-1$	$2s_3-1$	$2s_2$	$2s_3$	-1	3
(iii)	$2s_1-1$	$2s_2$	$2s_1$	$2s_3$	$2s_2-1$	$2s_3-1$	$+1$	6
(iv)	$2s_1-1$	$2s_2-1$	$2s_1$	$2s_3-1$	$2s_2$	$2s_3$	$+1$	6

From Table I, the value of the terms in (3.5) directly dependent on $P(1),\ldots, P(6)$, i.e., the value of

$$\left(\sum_{k=1}^{3} w_k^{\gamma_{P(2k-1)}-\gamma_{P(2k)}-1} f_{P(2k-1),P(2k)}\right) \delta_{P(5)+P(6),M-1} \times \prod_{l=1}^{3} \delta(\gamma_{2s_l}+\gamma_{2s_l-1}, M-1) \tag{3.9}$$

can be computed. The procedure can be made more efficient by ignoring some terms which will not contribute in the thermodynamic limit. The value of (3.5) for each entry in Table I is then

$$\text{(i)\&(ii)} \qquad F_1(\gamma_2, \gamma_4) = -2A_1(\gamma_2)\, A_2(\gamma_4) \tag{3.10}$$

where

$$A_1(\gamma) = a_2(\gamma) + [a_1(\gamma)]^2 \tag{3.11}$$

$$A_2(\gamma) = 4[a_1(\gamma)]^2\, a_2(\gamma) \tag{3.12}$$

$$\begin{aligned} \text{(iii)} \qquad F_2(\gamma_2, \gamma_4) = {} & 4\cos[2\pi(m_1-m_2)(\gamma_2-\gamma_4)/M] \\ & \times a_1(\gamma_4)[a_2(\gamma_4) + a_1(\gamma_2)\, a_1(\gamma_4)] \\ & \times [a_2(\gamma_4)\, a_1(\gamma_2) + a_2(\gamma_2)\, a_1(\gamma_4)] \end{aligned} \tag{3.13}$$

$$\begin{aligned} \text{(iv)} \qquad F_3(\gamma_2, \gamma_4) = {} & 4\cos[2\pi(m_1-m_2)(\gamma_2-\gamma_4)/M] \\ & \times a_1(\gamma_4)\, a_2(\gamma_4)[a_1(\gamma_4)\, a_1(\gamma_2) \\ & + a_2(\gamma_2)][a_1(\gamma_2) + a_1(\gamma_4)] \end{aligned} \tag{3.14}$$

Taking note of the factors (3.7), (3.8), and the multiplicities listed in Table I, one finds that the contribution to (3.5) is thus

$$\begin{aligned} & \frac{1}{\Xi_4} 6(2^{-2N}) \left(\frac{2\pi}{L}\right)^{4N} (-1)^N M^{2N-2}(N-2) \binom{N}{3} \sum_{\gamma_2=0}^{M-1} \sum_{\substack{\gamma_4=0 \\ \gamma_2 \neq \gamma_4}}^{M-1} \sum_{j=1}^{3} F_j(\gamma_2, \gamma_4) \\ & \quad \times \sum_{\substack{0 \leqslant \gamma_8, \gamma_{10}, \ldots, \gamma_{2N} \leqslant M-1 \\ \neq \gamma_2, \gamma_4, M-1-\gamma_4}} \ \sum_{P(2k) > P(2k-1)} \varepsilon(P) \left(\prod_{l=4}^{N} \delta_{P(2k)+P(2k-1)+1, M-1}\right) \end{aligned} \tag{3.15}$$

The problem of evaluating the contribution of (3.15) to (3.5) and (3.4) is now of a similar type to that encounted and solved⁽¹⁾ in the evaluation of Ξ_4. The techniques detailed in I allow us to evaluate the contribution of (3.15) to the distribution function (3.4) in the thermodynamic limit as

$$64\xi^3\int_0^1\int_0^1 dt\,ds\,\frac{1-4\xi B_1(t)}{y(t;\xi)\,y(s;\xi)}\{B_1(t)\,B_2(s)-4\cos[2\pi(m_2-m_1)(t-s)]$$
$$\times\{b_1(s)[b_2(s)+b_1(t)\,b_1(s)][b_2(s)\,b_1(t)+b_2(t)\,b_1(s)]$$
$$+b_1(s)\,b_2(s)[b_1(s)\,b_1(t)+b_2(t)][b_1(t)+b_1(s)]\}\} \qquad (3.16)$$

The functions B_1, B_2, b_1, and b_2 are continuous versions of A_1, A_2, a_1, and a_2, respectively, obtained by dividing through by M^2, M^4, M, and M^2, respectively, calling γ/M the variable t (or s, as appropriate) and ignoring terms $O(1/M)$. Explicitly,

$$\begin{aligned} &b_1(t)=t-1/2, \qquad b_2(t)=t^2-t \\ &B_1(t)=b_2(t)+[b_1(t)]^2, \qquad B_2(t)=4[b_1(t)]^2\,b_2(t) \end{aligned} \qquad (3.17)$$

Also

$$\begin{aligned} y(t,\xi)&=[1-4\xi B_1(t)]^2-16\xi^2B_2(t) \\ &=1-8\xi B_1(t)+\xi^2 \end{aligned} \qquad (3.18)$$

The most crucial feature of the calculation outlined above is to list correctly all permutations in the class and the corresponding parities and multiplicities. For the ξ^2 class these are given by Table II, while for the ξ^4 class I have listed the permutations in Table III.

The entries in both tables are to include the interchange $(P(1), P(2)) \leftrightarrow (P(3), P(4))$.

Use of Tables II and III and the procedure outlined above to derive (3.16) allow us to derive the evaluation of the truncated distribution function

$$\rho^T_{+,+}(m_1,m_2)$$
$$= -16\xi^2\int_0^1\int_0^1 dt\,ds\,\frac{[1-4\xi B_1(t)][1-4\xi B_1(s)]}{y(t;\xi)\,y(s;\xi)}$$
$$\times\cos 2\pi(m_1-m_2)(t-s)\{[b_2(t)\,b_1(s)+b_2(s)\,b_1(t)][b_1(t)+b_1(s)]$$

Table II. The ξ^2 Class of Permutations for $\rho_{+,+}$

$P(1)$	$P(2)$	$P(3)$	$P(4)$	$\varepsilon(P)$	Multiplicity
$2s_2-1$	$2s_2$	$2s_1-1$	$2s_1$	$+1$	1
$2s_1$	$2s_2$	$2s_1-1$	$2s_2-1$	-1	1
$2s_1-1$	$2s_2$	$2s_1$	$2s_2-1$	$+1$	1

$$
\begin{aligned}
&+[b_2(t)+b_1(s)\,b_1(t)][b_2(s)+b_1(t)\,b_1(s)]\}\\
&-256\xi^3\int_0^1\int_0^1 dt\,ds\,\frac{1-4\xi B_1(t)}{y(t;\xi)\,y(s;\xi)}\cos 2\pi(m_1-m_2)(t-s)\\
&\times\{b_1(s)[b_2(s)+b_1(t)\,b_1(s)][b_2(s)\,b_1(t)+b_2(t)\,b_1(s)]\\
&+b_1(s)\,b_2(s)[b_1(s)\,b_1(t)+b_2(t)][b_1(t)+b_1(s)]\}\\
&-512\xi^4\int_0^1\int_0^1 dt\,ds\,\frac{\cos 2\pi(m_1-m_2)(t-s)}{y(t;\xi)\,y(s;\xi)}\\
&\times\{b_1(t)\,b_1(s)[b_2(s)\,b_1(t)+b_2(t)\,b_1(s)]^2\\
&+b_1(t)\,b_1(s)\,b_2(t)\,b_2(s)[b_1(t)+b_1(s)]^2\\
&+2b_1(t)\,b_1(s)\,b_2(s)[b_1(s)\,b_1(t)+b_2(t)]^2\}
\end{aligned}
\tag{3.19}
$$

3.2. Opposite Charges

Now we turn our attention to calculating

$$\rho_{+,-}\left(m_1, l_1-\frac{1}{2}\right)=\frac{1}{\Xi_4}\sum_{N=1}^{M}\zeta^{2N}Z_N\left(m_1, l_1-\frac{1}{2}\right) \tag{3.20}$$

where

$$Z_N(m_1, l_1-1/2)=\sum_{m_2,\ldots,m_N=1}^{M}\ \sum_{l_2,\ldots,l_N=1}^{M} W_{N4}/[(N-1)!]^2 \tag{3.21}$$

The sum over the l's and m's in (3.21) can be performed by substituting (3.1) in (3.21). They contribute

$$(-1)^{N-1}M^{2N-2}\prod_{l=2}^{N}\delta_{\gamma_{2l}+\gamma_{2l-1}+1,M}\,\delta_{P(2l)+P(2l-1)+1,M} \tag{3.22}$$

Table III. The ξ^4 Class of Permutations for $\rho_{+,+}$

$P(1)$	$P(2)$	$P(3)$	$P(4)$	$P(5)$	$P(6)$	$P(7)$	$P(8)$	$\varepsilon(P)$	Multiplicity
$2s_1$	$2s_2$	$2s_3$	$2s_4$	$2s_1-1$	$2s_2-1$	$2s_3-1$	$2s_4-1$	$+1$	3
$2s_1$	$2s_2$	$2s_3$	$2s_4$	$2s_1-1$	$2s_3-1$	$2s_2-1$	$2s_4-1$	-1	6
$2s_1-1$	$2s_2-1$	$2s_3-1$	$2s_4-1$	$2s_1$	$2s_2$	$2s_3$	$2s_4$	$+1$	3
$2s_1-1$	$2s_2-1$	$2s_3-1$	$2s_4-1$	$2s_1$	$2s_3$	$2s_2$	$2s_4$	-1	6
$2s_1$	$2s_2$	$2s_3-1$	$2s_4-1$	$2s_1-1$	$2s_2-1$	$2s_3$	$2s_4$	$+1$	6
$2s_1$	$2s_2-1$	$2s_3$	$2s_4-1$	$2s_1-1$	$2s_3-1$	$2s_2$	$2s_4$	-1	12

Table IV. The ξ Class of Permutations for $\rho_{+,-}$

$P(1)$	$P(2)$	$\varepsilon(P)$	Multiplicity
1	2	1	1

The permissible values of $P(1),\ldots, P(4)$ can be categorized into the classes ξ, ξ^2, ξ^3, and ξ^4. These values are given in Tables IV–VII (we take $s_3 > s_2 > s_1$).

Proceeding as in Section 3.1, we can use these tables to derive the result

$$\begin{aligned}
&\rho^T_{+,-}\left(m, l, -\frac{1}{2}\right)\\
&\quad= -4\xi \int_0^1 \int_0^1 dt\, ds \frac{\cos 2\pi(l+1/2-m)(t-s)}{y(t;\xi)\, y(s,\xi)}\\
&\qquad\times [1-4\xi B_1(t)][1-4\xi B_1(s)][b_2(t)+b_1(s)\, b_1(t)]\\
&\qquad -32\xi^2 \int_0^1 \int_0^1 dt\, ds \frac{\cos 2\pi(l+1/2-m)(t-s)}{y(t;\xi)\, y(s;\xi)} [1-4\xi B_1(s)]\\
&\qquad\times \{b_2(t)\, b_1(t)[b_1(t)+b_1(s)] + b_1(t)[b_2(t)\, b_1(s)+b_2(s)\, b_1(t)]\}\\
&\qquad -256\xi^3 \int_0^1 \int_0^1 dt\, ds \frac{\cos 2\pi(l+1/2-m)(t-s)}{y(t;\xi)\, y(s;\xi)}\\
&\qquad\times b_2(t)\, b_1(t)\, b_1(s)[b_2(s)+b_1(t)\, b_1(s)]
\end{aligned} \tag{3.23}$$

3.3. Simplified Expressions for the Truncated Distributions

In ref. 1, I showed that the zeros of Ξ are unchanged by the mapping

$$\xi \mapsto 1/\xi \tag{3.24}$$

Table V. The ξ^2 Class of Permutations for $\rho_{+,-}$

$P(1)$	$P(2)$	$P(3)$	$P(4)$	$\varepsilon(P)$	Multiplicity
1	$2s-1$	2	$2s$	-1	1
2	$2s$	1	$2s-1$	-1	1
$2s-1$	$2s$	1	2	1	1

Table VI. The ξ^3 Class of Permutations for $\rho_{+,-}$

$P(1)$	$P(2)$	$P(3)$	$P(4)$	$P(5)$	$P(6)$	$\varepsilon(P)$	Multiplicity
$2s_1$	$2s_2$	$2s_1-1$	$2s_2-1$	1	2	-1	2
$2s_1-1$	$2s_1$	1	$2s_2-1$	2	$2s_2$	-1	2
$2s_1$	$2s_2-1$	1	$2s_1-1$	2	$2s_2$	-1	2

This is a general property of a lattice gas in periodic boundary conditions, where

$$\xi = e^{-\beta E_0}\zeta \tag{3.25}$$

E_0 denoting the ground-state energy and ζ the fugacity. Suppose, furthermore, that in the thermodynamic limit the zeros form a one-dimensional region along the contour C in the complex plane with normalized density $f(w)$. Then the dimensionless particle density can be written

$$\tau\rho = \int_C dw\, f(w) \frac{-\xi(w+w^{-1})+2\xi}{1-\xi(w+w^{-1})+\xi^2} \tag{3.26}$$

where τ is the lattice spacing. It follows immediately from (3.26) that under the transformation (3.24) the particle density transforms as

$$\tau\rho \mapsto 1-\tau\rho \tag{3.27}$$

How do the truncated distribution functions transform under (3.24)? Cornu and Jancovici$^{(8)}$ have addressed this question for the truncated distribution functions obtained by Gaudin$^{(7)}$ for the two-dimensional generalization at the lattice gas considered here at $\Gamma=2$. They found that the truncated distributions were invariant under the transformation (3.24).

In the case of the distribution function between like charges this assumes that the particles are separated from each other. When two particles occupy the same position the distribution function vanishes and we have

$$\rho^T_{+,+}(0) = -(\tau\rho)^2 \tag{3.28}$$

Table VII. The ξ^4 Class of Permutations for $\rho_{+,-}$

$P(1)$	$P(2)$	$P(3)$	$P(4)$	$P(5)$	$P(6)$	$P(7)$	$P(8)$	$\varepsilon(P)$	Multiplicity
$2s_1$	$2s_2$	$2s_1-1$	$2s_2-1$	1	$2s_3-1$	2	$2s_3$	1	6

Hence, under the transformation (3.24), since the density transforms as (3.27) and the distribution function still vanishes,

$$\rho^T_{+,+}(0) \mapsto -(1-\tau\rho)^2 \tag{3.29}$$

An investigation of the transformation properties of the results (3.19) and (3.23) leads to substantially simplified expressions. Expanding the numerators in (3.19) and (3.23), collecting together terms with the same leading-order low-density behavior, and using (3.17) and (3.18) shows

$$\begin{aligned}\rho^T_{+,+}(m_1, m_2) = {} & -64\xi^2 \int_0^1 \int_0^1 dt\, ds \frac{b_1(t)\, b_2(t)\, b_1(s)\, e^{2\pi i(m_1-m_2)(t-s)}}{y(t;\xi)\, y(s;\xi)} \\ & - \left| \int_0^1 dt \frac{[4\xi B_1(t) - \xi^2]\, e^{2\pi i(m_1-m_2)t}}{y(t;\xi)} \right|^2 \end{aligned} \tag{3.30}$$

and

$$\begin{aligned}\rho^T_{+,-}&\left(l+\frac{1}{2}, m\right) \\ &= -4(\xi+\xi^3) \int_0^1 \int_0^1 dt\, ds \frac{b_2(t) + b_1(s)\, b_1(t)}{y(t;\xi)\, y(s;\xi)} e^{2\pi i(l+1/2-m)(t-s)} \\ &\quad + 8\xi^2 \int_0^1 \int_0^1 dt\, ds \frac{b_1(t)\, b_1(s) - b_2(t)}{y(t;\xi)\, y(s;\xi)} e^{2\pi i(l+1/2-m)(t-s)} \end{aligned} \tag{3.31}$$

where b_1, b_2, B_1, and y are defined by (3.17) and (3.18).

The first term of the expression (3.30) is clearly invariant under (3.24), while some algebraic manipulation shows that the second term is, too, provided $m_1 \neq m_2$. If $m_1 = m_2$, then this second term gives the properties (3.28) and (3.29). The truncated distribution function between unlike charges is invariant under the transformation (3.24), as can be seen immediately from (3.31).

4. SUM RULES

4.1. Compressibility Sum Rule

A good check on the accuracy of our working is the verification of some sum rules of general validity involving the two-particle distributions. One such identity is the compressibility sum rule, which here states

$$\sum_{m=-\infty}^{\infty} \rho^T_{+,+}(|m|) = \xi \frac{\partial \tau\rho_+}{\partial \xi} - \tau\rho_+ \tag{4.1}$$

where

$$\rho^T_{+,+}(|m|) \equiv \rho^T_{+,+}(m_1, m_2) \qquad m = m_1 - m_2 \tag{4.2}$$

To verify (4.1), substitute (3.30) in the left-hand side. Interchanging the order of summation and integration then gives

$$-16\xi^2 \int_0^1 dt \frac{B_2(t)}{[y(t;\xi)]^2} - \int_0^1 dt \frac{[4\xi B_1(t) - \xi^2]^2}{[y(t;\xi)]^2} \tag{4.3}$$

Since

$$\tau\rho_+ = \int_0^1 dt \frac{-4\xi B_1(t) + \xi^2}{y(t;\xi)} \tag{4.4}$$

and

$$B_2(t) = -\frac{1}{16} + [B_1(t)]^2 \tag{4.5}$$

we can readily identify (4.3) with the right-hand side of (4.1) as required.

From (4.4) and (3.18) the right-hand side of (4.1) diverges when $\xi = 1$ (the transition point). I shall show in Section 4.3 that the left-hand side diverges also. The compressibility is infinite at the transition point.

4.2. Perfect Screening Sum Rule

This sum rule, which is thought to be equivalent to the existence of the thermodynamic limit, states that the amount of charge contained within the screening cloud of any charge in the system is equal and opposite to that of the charge. Thus

$$-\sum_{l'=-\infty}^{\infty} \rho^T_{+,-}(|l'+1/2|) + \sum_{m=-\infty}^{\infty} \rho^T_{+,+}(|m|) = -\tau\rho_+ \tag{4.6}$$

where I have written

$$\rho^T_{+,-}(|l'+1/2|) \equiv \rho^T_{+,-}(l+1/2, m), \qquad l' = l - m \tag{4.7}$$

The sum rule can be checked by using the expressions (3.30) and (3.31) and interchanging the order of summation and integration. The resulting expression is to be compared to the right-hand side of (4.6) with (4.4) modified so that the denominator is $[y(t;\xi)]^2$. As expected, the sum rule is valid.

4.3. Asymptotic Behavior of the Truncated Distributions

I have previously noted(9) that for the log-potential lattice gas a sum rule can be derived to specify the state of the system as conducting or insultating. I briefly review the argument here.

A conducting state is characterized by its ability to screen an external charge density $\delta\rho_{\text{ext}} = \lambda e^{ikx}$ in the long-wavelength, $k \to 0$, limit. In the log-potential lattice gas this means that the change in charge density on neighboring lattice sites (one available to positive charges, the other to negative charges) will exactly compensate the amount of charge within a region of length τ (the lattice spacing) in the external charge density.

A linear response argument shows that this characterization leads to the sum rule

$$2 \sum_{m=-\infty}^{\infty} [\rho_+ \delta_{m,0} + e^{ikm\tau} \rho^T_{+,+}(|m|) - e^{ik(m+1/2)\tau} \rho^T_{+,-}(|m+1/2|)] \\ \sim \tau\, |k|/(\pi\Gamma) \tag{4.8}$$

Use of the Poisson summation formula and the theory of Fourier transforms(10) shows that this is equivalent to the large-$|m|$ behavior

$$C^T_2(m)/q^2 \equiv 2[\rho^T_{+,+}(|m|) - \rho^T_{+,-}(|m|)] \sim -1/\pi^2\Gamma\, |m|^2 \tag{4.9}$$

[In ref. 9, I erroneously wrote $\rho^T_{+,-}(|m|)$ as $\rho^T_{+,-}(|m-1/2|)$. The correct form (4.9) is necessary whenever $\rho^T_{+,-}$ decays slower than $1/m^2$.] The large-$|m|$ expansion of $\rho^T_{+,-}(|m|)$ is to be interpreted as the large-l' expansion of $\rho^T_{+,-}(|l'+1/2|)$ with $l'+1/2$ replaced by m in the final expression.

It is a straightforward exercise in integration by parts to deduce from (3.30) and (3.31) the large-separation expansions

$$\rho^T_{+,+}(|m|) \\ \sim \frac{1}{(2\pi m)^4}\left(\frac{1024}{(1-\xi)^6} - \frac{3072}{(1-\xi)^5} + \frac{3200}{(1-\xi)^4} - \frac{1280}{(1-\xi)^3} + \frac{128}{(1-\xi)^2}\right) \\ + O(m^{-6}), \qquad \xi \neq 1 \tag{4.10}$$

$$\rho^T_{+,-}\left(\left|l'+\frac{1}{2}\right|\right) \\ \sim \frac{1}{[2\pi(l'+1/2)]^4}\left(\frac{1024}{(1-\xi)^6} - \frac{3072}{(1-\xi)^5} + \frac{3456}{(1-\xi)^4} - \frac{1792}{(1-\xi)^3} \right. \\ \left. + \frac{400}{(1-\xi)^2} - \frac{16}{1-\xi}\right) + O\left(\left|l'+\frac{1}{2}\right|^{-6}\right), \qquad \xi \neq 1 \tag{4.11}$$

$$\rho^T_{+,+}(|m|)$$
$$\sim \frac{1}{16\,|m|} - \frac{1}{16\pi^2 m^2} + O(|m|^{-3}), \qquad \xi = 1 \tag{4.12}$$

$$\rho^T_{+,-}\left(\left|l' + \frac{1}{2}\right|\right)$$
$$\sim \frac{1}{16\,|l' + 1/2|} + \frac{1}{16\pi^2\,|l' + 1/2|^2} + O\left(\left|l' + \frac{1}{2}\right|^{-3}\right), \qquad \xi = 1 \tag{4.13}$$

$$\rho^T_{+,+}(|m|)$$
$$\sim \frac{128\xi^2}{(2\pi m)^4}, \qquad \xi \to 0 \tag{4.14}$$

$$\rho^T_{+,-}\left(\left|l' + \frac{1}{2}\right|\right)$$
$$\sim \frac{16\xi}{[2\pi(l' + 1/2)]^4}, \qquad \xi \to 0 \tag{4.15}$$

Before discussing the results (4.10)–(4.15) in the context of the sum rule (4.9), I draw attention to some other features of the expansions. The leading-order coefficient as $\xi \to 1$ in (4.10) and the coefficient in (4.14) are positive. Since $\rho^T_{+,+}$ is decaying as m^{-4}, this implies that their Fourier transform for small k is also positive. This must be the case as the Fourier transform of $\rho^T_{+,+}$ can be written as the canonical average of a nonnegative quantity and is thus nonnegative. Similarly, the coefficient of $1/|m|$ in (4.12) must be positive. Another feature of (4.10) and (4.11) is that the coefficients of the $(1-\xi)^{-k}$ terms add to zero. This is consistent with the leading-order, small-ξ behavior given by (4.14) and (4.15).

Now consider the sum rule (4.9) and suppose $\Gamma = 2$. From (2.17) and (2.18) we see

$$C^T_2(m)/q^2 \sim -2\tau\rho_+(1 - \tau\rho_+)/\pi^2\,|m|^2 \tag{4.16}$$

and so (4.9) is obeyed only at the density $\tau\rho_+ = 1/2$. However, there is no nonanalytic behavior associated with the insulator–conductor transition. From (1.1), (1.4), and Fourier transform theory$^{(10)}$ we see from (4.16) that at $\Gamma = 2$

$$1/\varepsilon(k) \sim 1 - 4\tau\rho_+(1 - \tau\rho_+) \qquad \text{as} \quad k \to 0 \tag{4.17}$$

In the quantum Brownian motion problem mentioned in the Introduction, the mobility can be calculated exactly at the coupling corresponding to $\Gamma=2$,$^{(11)}$ and is also found to vary continuously with the dimensionless parameter.

Next consider the $\Gamma=4$ results (4.10)–(4.15). For $\xi \neq 1$ we see from (4.10) and (4.11) that the left-hand side of (4.9) decays as $O(1/|m|^4)$, so the system is insulating. In the low-density $\xi \to 0$ limit, from (4.14) and (4.15) we can deduce

$$1/\varepsilon(k) \sim 1-(32/3\pi^2)(\xi k^2) \qquad \text{as} \quad k \to 0 \tag{4.18}$$

On the other hand, from (4.12) and (4.13), if $\xi=1$ $(\tau\rho_+ = 1/2)$,

$$C_2^T(m)/q^2 \sim -1/4\pi^2\,|m|^2 \tag{4.19}$$

which from (4.9) says that the system is in a conducting state. Thus, the phase transition along the $\Gamma=4$ isotherm is of an insulator–conductor type, in which the insulating phase has a dielectric constant equal to unity.

5. DISCUSSION

5.1. Decay of the Correlations in the Low-Density Regime of the Insulating Phase

The truncated two-particle distributions are functions of the scaled fugacity ξ and the particle separation. In the insulating regime we expect $\rho^T_{+,+}$ and $\rho^T_{+,-}$ to be analytic functions of ξ. A feature of the exact results (2.10), (2.18), (4.14), and (4.15) is that the leading-order behavior as a function of ξ is $O(\xi^2)$ for $\rho^T_{+,+}$ and $O(\xi)$ for $\rho^T_{+,-}$. The former result is readily understood as a consequence of expanding both sides of the compressibility sum rule (4.1) in powers of ξ. The first nonzero term of the right-hand side will be $O(\xi^2)$. According to the perfect screening sum rule (4.6), one can now conclude that for small ξ, $\rho^T_{+,-}$ behaves as $O(\xi)$.

In the conducting regime the correlations are not expected to be analytic functions of ξ around $\xi=0$. This would be a consequence of the zeros of the grand partition function, which are conjectured to be on the negative real axis throughout the region, pinching the origin in the thermodynamic limit.

It was remarked in the Introduction that for the dielectric phase one expects in general

$$\hat{C}^T(k) \sim o(|k|) \tag{5.1}$$

which is the same as saying

$$C(y) \sim o(1/y^2) \tag{5.2}$$

Fig. 1. A typical configuration required for the heuristic derivation of (5.3). The fixed charges are labeled A and B. The dots denote empty lattice sites available to the positive charges, while the crosses denote empty lattice sites available to the negative charges.

In fact, analogous to the Kosterlitz–Thouless presumption$^{(12)}$ for the two-dimensional generalization of this system, we might suppose that for low densities and the dielectric regime ($\Gamma \geqslant 2$)

$$\rho^T_{+,-}(y) \sim \zeta^2 e^{-\Gamma \log y} = \zeta^2 y^{-\Gamma} \tag{5.3}$$

Indeed, the exact results (2.18) and (4.15) are in agreement with (5.3) (up to a numerical factor).

Although there is thus some evidence in favor of the validity of (5.3), I believe there is a fundamental problem with the reasoning leading to its formulation. Recall the derivation of (5.3). Consider a positive and negative charge at some fixed distance y in the insulating regime at low density. Since by assumption the system is dielectric and behaving as a vacuum, the particles interact via their bare (dimensionless) potential $-\Gamma \log r$ (see Fig. 1). Furthermore, since the system is assumed dilute ($\zeta \to 0$), one need only consider the first term (Boltzmann factor of the fixed charges) in the grand canonical ensemble expression for the distribution function. Thus the result (5.3).

The problem with this argument is the neglect of the fact that the positive and negative charges are themselves part of the system. As such they will each be one member of a positive–negative charge combination (dipole), so the effective pair potential will not be $-\Gamma \log r$ (see Fig. 2). Thus it seems that the statement (5.3) requires further study.

5.2. Scaling Properties

Of fundamental importance in the general theory of phase transitions is the behavior of the correlation length near the critical point. From (4.10) and (4.11) one sees that the length scale of the truncated distribution functions is

$$m\,|1-\xi|^{3/2} \tag{5.4}$$

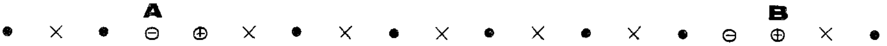

Fig. 2. An actual typical configuration in the insulating regime. The charges A and B are each part of a dipole.

as $\xi \to 1$. However, the charge–charge correlation exhibits a different length scale. From (4.10) and (4.11) we see

$$\frac{C_2^T(m)}{q^2} \sim -\frac{1}{(2\pi m)^4}\left(\frac{512}{(1-\xi)^4}-\frac{1024}{(1-\xi)^3}+\frac{544}{(1-\xi)^2}-\frac{32}{1-\xi}\right) \tag{5.5}$$

and thus the length scale is

$$m\,|1-\xi| \tag{5.6}$$

5.3. Phase Diagram

The results of Section 4.3 regarding the nature of the phase at $\xi = 2$ and 4 and the conjectures in I concerning the zeros of the grand partition function strongly suggest the phase diagram given in Fig. 3. The line $\Gamma = 2$ is the boundary between the conducting and insulating phases, the characteristics of this isotherm having been discussed in Section 4.3.

The phase diagram bears a striking resemblance to that of the one-component log-gas in a periodic potential in one dimension. With μ

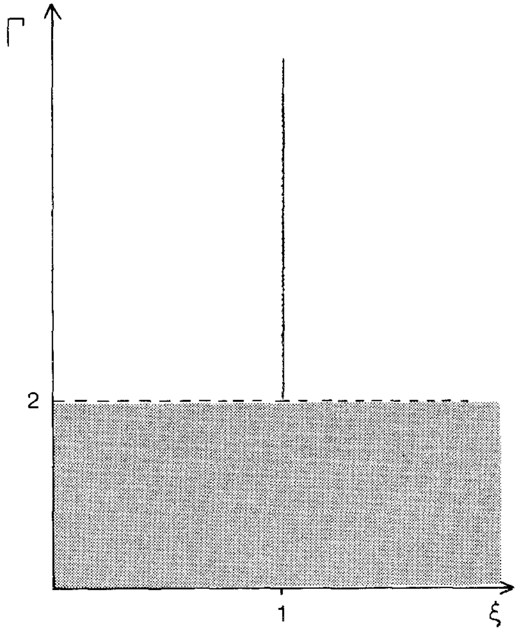

Fig.3. Conjectured phase diagram in the ξ–Γ plane. The shaded region is conducting, along with the entire line $\xi = 1$.

denoting the particle density, the phase diagram has been conjectured for an arbitrary background potential of period $1/\mu$.(5) If the coupling of the particle at point y to the background has a Boltzmann factor of the form

$$f(y) \equiv \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n \mu y} \tag{5.7}$$

then the system undergoes a conducting–insulating transition at $\Gamma = 2$ independent of the particular $f(y)$ ($\Gamma < 2$ is the conducting regime, $\Gamma \geqslant 2$ is the insulating regime). The averaged quantity

$$S(y) \equiv \mu \int_0^{1/\mu} dy' \, \rho^T(y+y', y') \tag{5.8}$$

plays the role of $C^T(y)/q^2$, with a conducting phase being characterized by the leading-order nonoscillatory behavior

$$S(y) \sim -\frac{1}{\pi^2 \Gamma y^2} \tag{5.9}$$

This sum rule is conjectured to hold for all potentials (5.7) provided $\Gamma < 2$, just as the two-component system is conjectured to be conducting for all ξ and $\Gamma < 2$.

At $\Gamma = 2$ in the one-component system the leading-order nonoscillating behavior is given by

$$S(y) \sim -\frac{1-(c_1/c_0)^2}{2(\pi y)^2} \tag{5.10}$$

so that the system is conducting if $c_1 = 0$. However, there is no nonanalytic behavior associated with this transition. The behavior is analogous to that of the two-component system at $\Gamma = 2$ with $(1-\xi)$ identified with c_1/c_0.

At $\Gamma = 4$, the leading-order nonoscillatory behavior in the one-component system is given by

$$S(y) \sim \begin{cases} b/y^4, & c_1 \neq 0 \\ -1/(4\pi^2 y^2), & c_1 = 0 \end{cases} \tag{5.11}$$

where b is a nonzero constant. The thermodynamic quantities exhibit a singularity as $c_1 \to 0$. The singular part of the dimensionless free energy βf behaves as

$$\beta f_{\text{sing}} \sim \tfrac{1}{2}(c_1/c_0)^2 \log |c_1| \tag{5.12}$$

and the singular part of the density $\rho(y)$ behaves as

$$\rho_{\text{sing}}(y) \sim -\frac{\mu f(y)}{(c_0)^2} \cos(2\pi\mu y)\, c_1 \log |c_1| \tag{5.13}$$

From Section 4.3 we know the two-component system has the behavior (5.11) again with $1-\xi$ identified with c_1/c_0. This also holds for the singular behavior (5.12) and (5.13), since from I in the two-component system

$$\beta(P-P_c) \sim -\frac{(1-\xi)^2}{8\tau} \log |1-\xi|, \qquad P_c = \frac{1}{\tau}(2\log 2 - 1) \tag{5.14}$$

and

$$\rho_+ - \rho_c \sim \frac{1}{4\tau}(1-\xi) \log |1-\xi|, \qquad \rho_c = \frac{1}{2\tau} \tag{5.15}$$

The correspondence between c_1/c_0 and $1-\xi$ cannot continue throughout the insulating regime. The conjectured phase diagram$^{(5)}$ of the one-component system has the region $18 > \Gamma > 8$, $c_1 = 0$ as an insulator unless both c_1 and c_2 vanish. In fact, for the values of Γ in the range

$$2p^2 < \Gamma < 2(p+1)^2 \tag{5.16}$$

(p an integer) a previous heuristic argument$^{(5)}$ predicts a conducting phase only if

$$c_1, c_2, \ldots, c_p = 0 \tag{5.17}$$

Of course, our understanding of both systems would be enhanced if the correspondence, and its breakdown for $\Gamma > 8$, could be adequately explained.

Addendum. M. L. Rosinberg has informed me that he has succeeded in generalizing Gaudin's$^{(7)}$ method to obtain a simplified derivation of (3.30) and (3.31).

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The remarks of the referess have been included in the revised version of this work.

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